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Spin–Spin Correlation Function in the Two-Dimensional Ising Model with Linear Defects. I. $T < T_c$

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The dispersion expansion for the spin correlation function in the two-dimensional Ising model with linear defects below T_c is derived. The asymptotic behavior is computed by a steepest descent analysis. The lattice is divided into four domains with different asymptotic behaviors. In particular, the correlation length inside certain domains is a function of the defect.

KEY WORDS: 2D spin system; Ising model; linear defect; spin correlation.

1. INTRODUCTION

The simplest way to modify a homogeneous two-dimensional Ising model, i.e., one with all horizontal and all vertical interactions equal to E_1 and E_2, to include a row of defect is either to modify a row of horizontal bonds to E'_1 or change a row of vertical bonds to E'_2. We call the former a line-defect model and the latter a ladder-defect model. Both models contain a marginal operator,[1,2] because the scaling dimension of the perturbation equals the scaling dimension of the energy-density operator.

Fisher and Ferdinand[3] first studied the incremental critical specific heat for the linear defect models. Bariev[2] computed the local magnetizations for both models. He obtained the defect-dependent critical exponent β for local magnetization near the defect,

$$\beta = (\arccos \tau)^2/(2\pi^2) \tag{1.1}$$

$$\tau = \begin{cases} \tanh 2(H'_c - H_c), & \text{line defect model} \\ (\cosh 2V'_c - \cosh 2V_c)(\cosh 2V'_2 \cosh 2V_2)^{-1}, & \text{ladder defect model} \end{cases} \tag{1.2}$$

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where $H_c = E_1/kT_c$, etc. McCoy and Perk[4] calculated the two-spin correlation function along the defect row for the line-defect model. They found the critical exponent $\eta = 2\beta$, where β is the same as in (1.1). The energy-density correlations for the line-defect model have been computed by Ko *et al.*[5] in the scaling limit; they found that the behavior at the critical point is similar to the electrostatic problem of two point charges near a boundary separating two media with different dielectric constants.

That τ depends on E_2' for the ladder-defect model is not unexpected, since when $E_2' = 0$ and $E_2' = E_2$, the ladder-defect model reduces to the semi-infinite free boundary system and to the infinite system, respectively, and the local critical exponents of these two systems are known to be different. In fact, the ladder-defect model is particularly interesting for being the simplest generalization that includes the infinite and the semi-infinite models. Many exact results are known about these two systems. For the infinite system, there has been a development that relates correlation functions to Painlevé functions[6]; while the semi-infinite system, with suitable boundary conditions, has led to an important understanding of many surface properties.[7] In addition, both infinite and semi-infinite models have been successfully treated at criticality by conformal algebra methods.[8] One naturally asks, what happens in between?

The line-defect and the ladder-defect models are intimately related. A duality transformation[9] that transforms the horizontal bonds to the vertical bonds, the order variables to the disorder variables, and vice versa takes one model to the other. But except for the less involved objects, such as energy-density correlations, the duality transformation does not provide information for order variables in one model from that in the other. Looking from a different angle, we may also say that knowledge about the behavior of order variables in both models enables one to gain more understanding for either model. Hence we propose to study in this paper a model that includes both the line-defect and the ladder-defect models as special cases. In particular we study the two-point spin correlations at arbitrary locations.

The Hamiltonian for this model is

$$-\frac{E}{kT} = \sum_{m,n} (E_1 \sigma_{m,n} \sigma_{m,n+1} + E_2 \sigma_{m,n} \sigma_{m+1,n}) + \sum_n \left[(E_1' - E_1) \sum_{m=-1/2}^{1/2} \sigma_{m,n} \sigma_{m,n+1} + (E_2' - E_2)\, \sigma_{-1/2,n} \sigma_{1/2,n} \right] \tag{1.3}$$

where the summations are over $n \in Z$, $m \in Z + \frac{1}{2}$. The boundary conditions are periodic, hence we do not consider the mixed-phase case.[10] This model reduces to the line-defect model with a row of modified horizontal interactions $2E_1'$ as $E_2' \to \infty$; when $E_1' = E_1$, it becomes the ladder-defect model. In the latter limit, we also have the half-plane case when $E_2' = 0$, and the pure case when $E_2' = E_2$. Figure 1 shows a portion of the lattice. Spin $\sigma_{m,n}$, which takes ± 1, sits on site (m, n), as indicated in the figure. We let m take half-integer values for simplicity of presentation.

We consider the case for $T < T_c$ in this paper and give the $T > T_c$ case in a subsequent paper.

In ref. 5, the scaling function of energy-density correlation for the line-defect model was expressed in a quadratic form of functions related to the Bessel functions. But it is known that spin correlation functions for Ising models have infinite dispersions, and we compute the dispersion expansions for the linear-defect model. The expansion below T_c is given in Eq. (3.5); in Eq. (4.8) we show the scaling form. Comparing to the pure case, we see that the expansion for the defect model has a discontinuity in functional form as one spin crosses the defect rows. The nature of this discontinuity can better be seen when it is expressed as a Fredholm determinant: When both spins are on the same side, it is the determinant of a system of four integral equations; but if one spin crosses the defect, the number is reduced from four to two. The special cases of the pure system and the half-plane system[2] can be expressed as the Fredholm determinant of a single equation. The relation with Fredholm equations will be given in a separate paper.

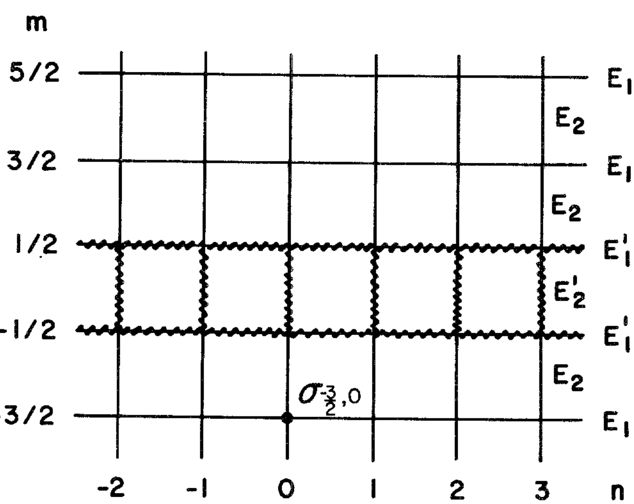

Fig. 1. A portion of the two-dimensional Ising lattice with nearest neighbor interactions E_1 and E_2 and with modified bonds E_1' and E_2' in the center rows.

In the pure system, two-point correlation functions can be specified by the separation between the two points alone, because of translational invariance. In the scaling limit, the angle dependence is further lost and rotational invariance becomes valid. For the defect model, clearly one no longer has full translational symmetry; nor does one obtain rotational symmetry in the scaling limit. In general, three coordinates are required, the horizontal and vertical distances between the spins, x and y, and the distance of one spin from the defect row, y_1. It is often convenient to use the separation r and the image separation $\bar{r}$. The image separation is the distance between one spin and the mirror image, with respect to the defect rows, of the other. This distance appears naturally already in the energy-density correlation in ref. 5, and apparently is closely related to the similarity with electrostatics.

We found that in the scaling limit, the two defects E_1' and E_2' appear only in the combination

$$\tau = \text{sgn}(T - T_c) \tanh[2(2E_1' - E_1 - E_2'^*)/kT_c] \tag{1.4}$$

This function is shown in Fig. 2.

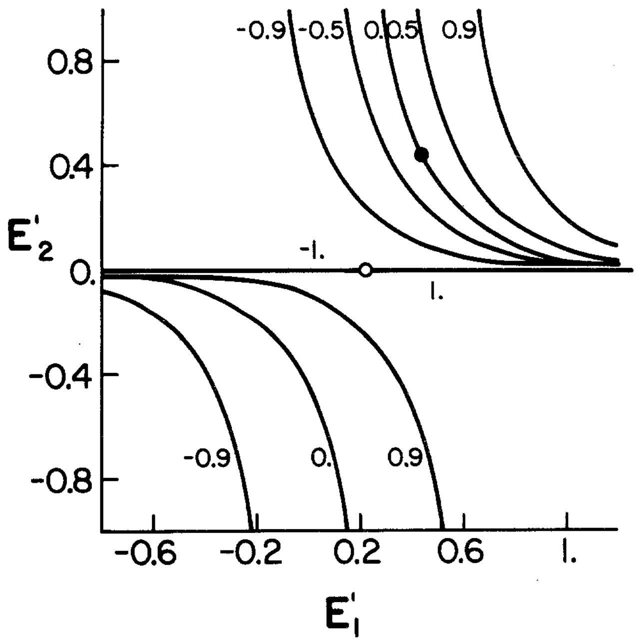

Fig. 2. Contours of $\text{sgn}(T - T_c)\tau$ in the defect-coupling space. The couplings are in units of kT_c and the bulk couplings are taken to be isotropic, $E_1 = E_2 \cong 0.44$. (●) The pure infinite system; the line $E_2' = 0$ gives the semi-infinite systems; (○) the point about which the function is antisymmetric.

We calculate the asymptotic behavior in the scaling limit as the two spins are far from one another and from the defects. These are given in Eqs. (5.5)–(5.10). The two cases for the spins on the same side and for spins on opposite sides of the defects are distinct. The latter case is much simpler; in fact, translational invariance is restored asymptotically. It is found that if $\tau > 0$, the correlation length become defect dependent in some regions. Fix one spin; then the domain for the locations of the second spin where a defect-dependent correlation length can be found is shown in the shaded portion of Fig. 3. In addition, in this region the power law changes to $r^{-1/2}$ from the usual r^{-2}, i.e., it becomes Ornstein–Zernike like. In the other regimes the effect of the defect is mainly a multiplicative factor depending on the defect and geometry. Figure 4 shows the most dominant contribution in the defect-inclination space for the case where the spins are on opposite sides.

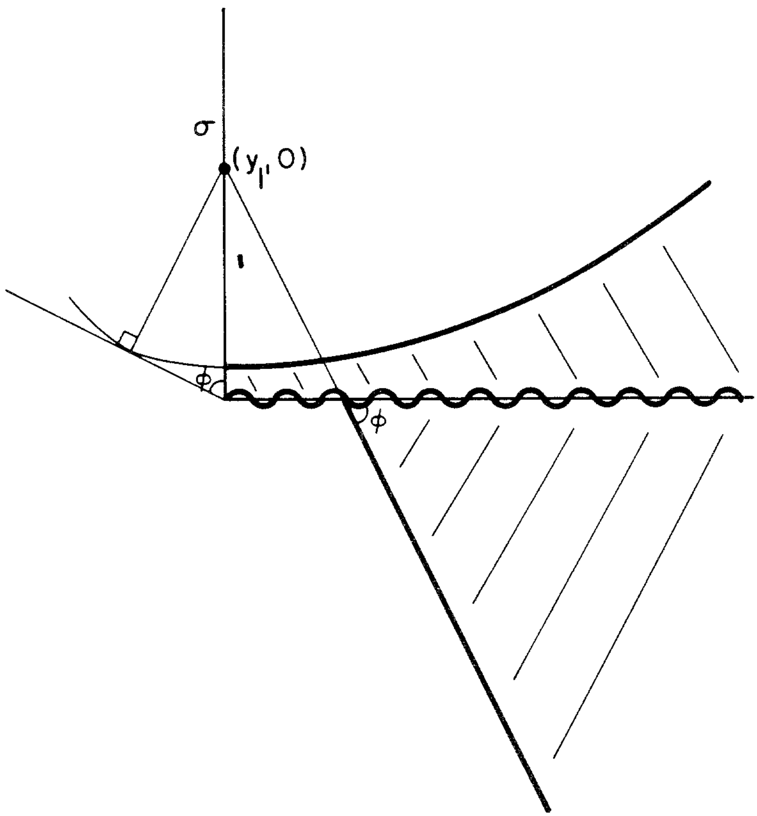

Fig. 3. If $\tau > 0$ and the second spin $\sigma_{y_2,x}$ is located in the shaded region or in its reflection, the correlation length is defect dependent. The wavy line represent the defects; the angle $\phi = \arcsin \tau$; and the curved line is part of a parabola.

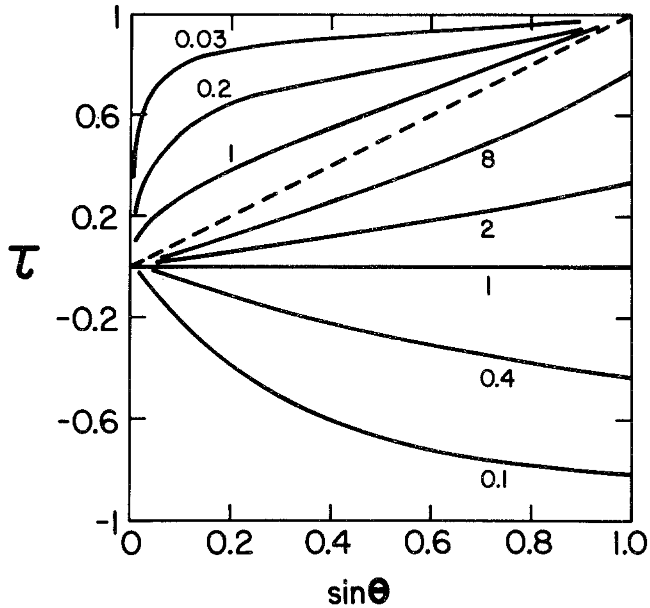

Fig. 4. Contours of the asymptotic two-point spin correlation for $T \to T_c^-$ as a function of defect τ and inclination, $\sin\theta = y/r$. The values given are obtained after normalizing by the bulk correlation below the broken line and by $\exp(-r - y\tau - x\bar{\tau})/(2\pi r)^{1/2}$ above.

2. FORMULATION OF THE PROBLEM

There exists different methods[11] for computing correlation functions in the Ising model. Here the method of Onsager and Kaufman using Clifford operators is adopted. We concentrate on the spin–spin correlation function, which can be expressed as a Toeplitz determinant,

$$\langle \sigma_{l,0}\sigma_{m,n}\rangle = \det{}^{1/2}\begin{pmatrix} A_0 + A_{11} & A_{12} \\ A_{21} & A_0 + A_{22} \end{pmatrix} \tag{2.1}$$

where the submatrices are of infinite size and will be defined shortly. The relation with Clifford operators is given in Appendix B of a companion paper. The block structure[6] of (2.1) is a result of our considering spins at arbitrary rows, and will not appear if we consider the special case $l = m$.[4]. The diagonal blocks have a pure part A_0 and a defect part, A_{11} or A_{22}, which vanishes in the homogeneous model. Separating the pure part from the diagonal blocks has the advantage that the determinant can be expanded using known results for A_0, namely, the determinant and the inverse of A_0.

All the submatrices on the right-hand side of Eq. (2.1) are Toeplitz with 2×2 matrices as their entries. The matrix A_0, which determines the magnetization in the homogeneous model, is

$$(A_0)_{kj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e^{-i(k-j)\phi} \begin{pmatrix} 0 & C(\phi) \\ -C(\phi)^{-1} & 0 \end{pmatrix} \tag{2.2}$$

where $k, j = 0, 1, 2,...$ and $C(\phi)$ is given in Appendix A.

The function $C(\phi)$ can be factored: $C(\phi) = C_+(\phi)\, C_-(\phi)$, with $C_\pm(\phi)$ analytic and nonzero in $|e^{i\phi}| \lessgtr 1$. This is a canonical factorization valid if $\ln C(\phi)$ is continuous and periodic [$C(\phi)$ has index zero]—a condition satisfied for $T < T_c$ but not for $T > T_c$. Technically this is why the $T < T_c$ and $T > T_c$ cases are entirely different. It is known that when the index is zero, Szegö's theorem and the Wiener–Hopf method can be applied to give[6,11]

$$\det A_0 = M^2 \tag{2.3}$$

and

$$\sum_{k,j=0}^{\infty} (A_0^{-1})_{kj}\, e^{i(k\phi - j\theta)} = \frac{a^+(\phi)\, a^-(\theta)}{1 - e^{i(\phi - \theta + i\varepsilon)}} \tag{2.4}$$

where $\varepsilon \to 0^+$, M is the spontaneous magnetization for the pure system, and

$$a^+(\phi) = \begin{pmatrix} 0 & -C_+(\phi) \\ C_+(\phi)^{-1} & 0 \end{pmatrix}, \quad a^-(\phi) = \begin{pmatrix} C_-(\phi)^{-1} & 0 \\ 0 & C_-(\phi) \end{pmatrix} \tag{2.5}$$

The other submatrices besides A_0 in (2.1) have the general form

$$(A_{pq})_{kj} = \sum_{s=1}^{2} (A_{pq}^s)_{kj} \tag{2.6a}$$

with

$$(A_{pq}^s)_{kj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e^{-i(k-j)\phi}\, X_{pq}^s(\phi)\, b_{pq}^s(\phi) \tag{2.6b}$$

where $p, q = 1, 2$ and $k, j = 0, 1, 2,....$ The functions $X_{pq}^s(\phi)$ may contain the defect strength κ. [Both $X_{pq}^s(\phi)$ and κ are given in Appendix A.] The 2×2 matrix $b_{pq}^s(\phi)$ has the simple factored form

$$b_{pq}^s = \begin{pmatrix} C_- \\ (-1)^{ps}(-iC_+^{-1}) \end{pmatrix} ((-1)^{qs}(iC_-^{-1}), C_+) \tag{2.7}$$

A closer look at the functions $X^s_{pq}(\phi)$ in (A.4) shows that not all combinations of (p, q, s) in Eq. (2.6) are relevant, because it is zero for $s=2$, which means s is redundant, if the spins are on opposite sides of the defects $(l<0<m)$. It is possible to eliminate the superscript for the other case $(0<l<m)$ also by introducing a more involved $b_{pq}(\phi)$; this is shown in ref. 5, and will not be presented here.

3. DISPERSION EXPANSION

As noted above, we want to compute the determinant in Eq. (2.1) by expanding around A_0. Using a well-known identity[12] for determinant and trace, it is easy to show from (2.1) that

$$\begin{aligned} \langle \sigma_{l,0}\sigma_{m,n}\rangle &= \det(A_0)\exp[\tfrac{1}{2}\operatorname{Tr}\ln(1+R)] \\ &= M^2 \exp\left(-\sum_{k=1}^{\infty} \frac{(-1)^k}{2k} \operatorname{Tr} R^k\right) \end{aligned} \tag{3.1}$$

where

$$R = \begin{pmatrix} A_0^{-1}A_{11} & A_0^{-1}A_{12} \\ A_0^{-1}A_{21} & A_0^{-1}A_{22} \end{pmatrix} \tag{3.2}$$

The first step in the evaluation of the trace over R^k is to write out the following:

$$\operatorname{Tr} R^k = \sum_{\{p_j, s_j\}} \sum_{\{n_i\}} \operatorname{tr} \prod_{j=1}^{k} [(A_0^{-1})_{n_{2j-2}, n_{2j-1}} (A^{s_j}_{p_j p_{j+1}})_{n_{2j-1}, n_{2j}}] \tag{3.3}$$

where the summations are over $p_j = 1,\ 2,\ s_j = 1,\ 2,\ j = 1,..., k$, with $p_{k+1} = p_1$, and over $n_i = 0,..., \infty$, $i = 1,..., 2k$, with $n_0 = n_{2k}$. The tr denotes the trace operation over the remaining 2×2 matrices.

Substituting (2.6) in (3.3), we find combinations of terms in the form of the left-hand side of Eq. (2.4); using it, we find that Eq. (3.3) simplifies. This leads to an expression containing the product

$$\operatorname{tr} \prod_{j=1}^{k} [a^-(\phi_j)\, b^{s_j}_{p_j p_{j+1}}(\phi_j)\, a^+(\phi_j)] \tag{3.4}$$

The above can be evaluated explicitly from Eqs. (2.5) and (2.7). Because b^s_{pq} has the factored form in (2.7), the algebra is tremendously simplified.

After some algebra and substitution, we find the dispersion expansion to be

$$\langle \sigma_{l,0}\sigma_{m,n}\rangle = M^2 \exp\left[-\sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{-1}{2\pi}\right)^k \sum_{\{p_j, s_j\}} \times \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_k K_{p_1 p_2}^{s_1 s_2}(\phi_1, \phi_2) \times K_{p_2 p_3}^{s_2 s_3}(\phi_2, \phi_3) \cdots K_{p_k p_1}^{s_k s_1}(\phi_k, \phi_1)\right] \tag{3.5a}$$

with

$$K_{pq}^{st}(\phi, \theta) = \frac{X_{pq}^{t}(\theta)}{1 - e^{i(\phi - \theta + i\varepsilon)}}\left[1 - (-1)^{p(s+t)}\frac{C_+}{C_-}(\phi)\frac{C_-}{C_+}(\theta)\right] \tag{3.5b}$$

Equation (3.5) has the form of a Fredholm determinant of a system of integral equations; this will be discussed elsewhere. Since X_{pq}^t decays exponentially with the distances [see (A.4)], the importance of each term in (3.5a) decreases with increasing k in general. However, at the critical temperature the exponent in X_{pq}^t vanishes and the expansion diverges.

4. CONTINUUM LIMIT

It is well known that in the homogeneous model there is a transition from an order to a disorder phase. This occurs at a critical temperature T_c given by

$$\sinh\left(\frac{2E_1}{kT_c}\right)\sinh\left(\frac{2E_2}{kT_c}\right) = 1 \tag{4.1}$$

The defect model has a phase transition occurring at the same T_c given above. We can see this from a transfer-matrix argument as follows. Let $\mathbf{T}$ be the row-to-row transfer matrix for the homogeneous model and let $\mathbf{T}'$ be the transfer matrix of the defect row. The partition functions for the homogeneous and the defect systems Z_h and Z_d can be expressed in terms of the transfer matrices as

$$Z_h = \operatorname{Tr} \mathbf{T}^N \tag{4.2}$$

$$Z_d = \operatorname{Tr} \mathbf{T}' \mathbf{T}^{N-1} \tag{4.3}$$

where N is the total number of rows. Let $\{\lambda_\alpha\}$ and $\{|\alpha\rangle\}$ be the set of eigenvalues and eigenfunctions for **T**. In terms of them Eqs. (4.2) and (4.3) become

$$Z_h = \lambda_0^N \left[1 + \left(\frac{\lambda_1}{\lambda_0}\right)^N + \left(\frac{\lambda_2}{\lambda_0}\right)^N + \cdots \right] \tag{4.4}$$

$$Z_d = \langle 0 | \mathbf{T}' | 0 \rangle \lambda_0^{N-1} \left[1 + \frac{\langle 1 | \mathbf{T}' | 1 \rangle}{\langle 0 | \mathbf{T}' | 0 \rangle} \left(\frac{\lambda_1}{\lambda_0}\right)^{N-1} + \cdots \right] \tag{4.5}$$

where λ_0 is the largest eigenvalue of **T**.

In the thermodynamic limit, $N \to \infty$, every term other than the first vanishes in (4.4). The same is true with (4.5) as long as $\langle 0 | \mathbf{T}' | 0 \rangle$ does not vanish. Hence the critical temperatures for both systems are determined by the singularity of λ_0.

In the next section, we shall restrict to the continuum, or scaling, limit. Let $T \to T_c$; the correlation length ξ diverges in this limit; consider large distance $R \to \infty$, keeping the scaled distance $r \sim R/\xi$ fixed. More precisely, for Eq. (3.5), we scale the parameters as follows:

$$x = \frac{n}{\xi_h}, \quad y_1 = \frac{|l|}{\xi_v}, \quad y_2 = \frac{m}{\xi_v}, \quad y = \frac{m-l}{\xi_v}, \quad \bar{y} = \frac{m+l}{\xi_v}, \quad \phi = \frac{w}{\xi_h} \tag{4.6}$$

where ξ_h and ξ_v are the horizontal and vertical correlation lengths, respectively. They diverge linearly as $T \to T_c$,

$$\begin{aligned} \xi_h^{-1} &= 2[V_c + H_c/\sinh(2H_c)]\, |T/T_c - 1| \\ \xi_v^{-1} &= \sinh(2H_c)\, \xi_h^{-1} \\ H_c &= E_1/kT_c, \qquad V_c = E_2/kT_c \end{aligned} \tag{4.7}$$

With the above scaling, it is straightforward to show that Eq. (4.5) becomes

$$\begin{aligned} \lim \langle \sigma_{l,0} \sigma_{m,n} \rangle = M_l M_m \exp \Bigg[-\sum_{k=2}^{\infty} \frac{1}{2k} \left(\frac{-i}{2\pi} \right)^k \sum_{\{p_j, s_j\}}{}' \\ \times \int_{-\infty}^{\infty} dw_1 \cdots \int_{-\infty}^{\infty} dw_k L_{p_1 p_2}^{s_1 s_2}(w_1, w_2) \\ \times L_{p_2 p_3}^{s_2 s_3}(w_2, w_3) \cdots L_{p_k p_1}^{s_k s_1}(w_k, w_1) \Bigg] \end{aligned} \tag{4.8}$$

with

$$L_{pq}^{st}(w, \rho) = \frac{Y_{pq}^t(\rho)}{(1+w^2)^{1/2}} \frac{(1+w^2)^{1/2} - (-1)^{p(s+t)} (1+\rho^2)^{1/2}}{w - \rho + i\varepsilon}$$

where $M_l = \langle \sigma_{l,0} \rangle$ is the local spontaneous magnetization at the lth row for the defect model, and the summations are over $p_j = 1, 2$, $s_j = 1, 2$, $j = 1, 2,..., k$, except that $p_1 = p_2 = \cdots = p_k$ is excluded. The kernal L^{st}_{pq} is the scaling limit of K^{st}_{pq} in (3.5b) up to some factors of $i\xi_h$. The function Y^t_{pq} can be obtained from X^t_{pq} by scaling as in (4.6) and taking the limit $\xi \to \infty$. This function is explicitly given in Appendix A. The convergence problem is not considered here; for that of the homogeneous system, see ref. 13.

The terms excluded from the exponential in (4.8), i.e., terms with all p_j equal, are those that do not vanish as $m - l \to \infty$. They come from the diagonal blocks of Eq. (2.1), and give exactly $M_l M_m / M^2$. Writing the expansion in the above form is more suitable for studying the asymptotic behavior, which is the subject of the next section.

5. ASYMPTOTIC DECAY

For operators with nonvanishing expectation values, we may consider the net correlation functions. The spin operator in the low-temperature phase is in this category. Let

$$\begin{aligned} &\langle \delta\sigma_{y_1,0}\, \delta\sigma_{y_2,x} \rangle \\ &\quad = \lim(\langle \sigma_{l,0}\sigma_{m,n} \rangle - \langle \sigma_{l,0} \rangle \langle \sigma_{m,n} \rangle)(M_l M_m)^{-1} \end{aligned} \tag{5.1}$$

The asymptotic decay for Eq. (4.8) can be computed from the first few terms in the infinite series on the right-hand side, because the kernel $L^{st}_{pq}(\omega, \rho)$ contains the function $Y^t_{pq}(\rho)$, which decays exponentially for large scaled distances. The leading behavior of these terms can be calculated by the method of steepest descent. The details of the calculation are given in Appendix B. Here we only present the final results.

The asymptotic decay for the homogeneous Ising model is[11]

$$\langle \delta\sigma_0\, \delta\sigma_r \rangle = e^{-2r}/8\pi r^2 \tag{5.2}$$

The above is not of the Ornstein–Zernike form, $e^{-r}/r^{1/2}$, due to the unique property of the low-temperature phase of the two-dimensional Ising model: the eigenstates of the transfer matrix are all even-particle states. Hence[14] the leading contribution cannot come from one-particle states to produce an OZ-type behavior.

For the defect model, the results are much more complicated in two respects: One is the very different behaviors for the two domains $l < 0 < m$ and $0 < l < m$; the other is the appearance of a defect-dependent correlation length, which is contained in the terms preceded by a step function below. The former feature is apparant from the beginning of our derivation through the definition of the function X^s_{pq}; the latter feature is found only as we study the asymptotic behavior of the correlation function.

Let τ be the defect strength in the scaling limit,

$$\tau = \operatorname{sgn}(T - T_c)\tanh[2(2E_1' - E_1 - E_2'^*)/kT_c] \tag{5.3}$$

where the asterisk denotes the dual,

$$\sinh\frac{2E_2'}{kT_c}\sinh\frac{2E_2'^*}{kT_c} = 1 \tag{5.4}$$

Since the defect model reduces to the homogeneous model for $\tau = 0$, we write

$$\langle \delta\sigma_{y_1,0}\,\delta\sigma_{y_2,x}\rangle_\tau = f_\tau(r, \bar{r}, y)\langle\delta\sigma_0\,\delta\sigma_r\rangle + \tau g_\tau(r, \bar{r}, y) \tag{5.5}$$

It is clear that $f_0(\cdots) = 1$. In general, the deviations f_τ and g_τ are the following:

(1) For $y_1 \ll 0 \ll y_2$,

$$f_\tau(r, \bar{r}, y) = \bar{\tau}^2\left(\frac{y}{r\tau - y}\right)^2 \tag{5.6}$$

$$g_\tau(r, \bar{r}, y) = \Theta\left(\tau - \frac{y}{r}\right)\frac{\bar{\tau}y}{r\tau - y} \times\left(\frac{r\bar{\tau} - x}{r\tau + y}\right)^2\frac{\exp(-r - y\tau - x\bar{\tau})}{(2\pi r)^{1/2}} \tag{5.7}$$

(2) For $0 \ll y_1 \ll y_2$,

$$f_\tau(r, \bar{r}, y) = 1 \tag{5.8}$$

$$\begin{aligned} g_\tau(r, \bar{r}, y) = &-\left[\frac{x}{\bar{r}\tau - \bar{y}}\,\frac{\bar{r} - r}{\bar{r} + r}\,\frac{e^{(-r-\bar{r})}}{2\pi(r\bar{r})^{1/2}} - \tau\left(\frac{x}{\bar{r}\tau - \bar{y}}\right)^2\frac{\exp(-2\bar{r})}{8\pi\bar{r}^2}\right. \\ &\left.+ (1-\tau)^{-1}\frac{ry}{(r+y)^2}\,\frac{\exp(-2r - 2y_1)}{(4\pi y_1)^{3/2}r^2}\right] \\ &-\Theta\left(\tau - \frac{\bar{y}}{\bar{r}}\right)\left[\frac{r\bar{\tau} - x}{r\bar{\tau} + x}\,\frac{\exp(-r - \bar{y}\tau - x\bar{\tau})}{(2\pi r)^{1/2}}\right. \\ &\left.-\frac{\tau x}{\bar{r}\tau - \bar{y}}\left(\frac{\bar{r}\bar{\tau} - x}{\bar{r}\tau + y}\right)^2\frac{\exp(-\bar{r} - \bar{y}\tau - x\bar{\tau})}{(2\pi\bar{r})^{1/2}}\right] \end{aligned} \tag{5.9}$$

where $\Theta(x)$ is the Heaviside step function, and

$$r^2 = x^2 + y^2, \qquad \bar{r}^2 = x^2 + \bar{y}^2, \qquad \bar{\tau}^2 = 1 - \tau^2 \tag{5.10}$$

The rather complicated expression in (5.9) is a clear indication that the breaking of rotational and translational invariance by the defect persists to the scaling regime. By comparison, the $l \ll 0 \ll m$ case is surprisingly simple; in particular, the translational invariance is restored in this asymptotic order.

Several observations can be made about the above results. First, if the term in g_τ containing the step function dominates, the correlation length is greater than in the bulk, and it depends on the defect as well as on the geometry. We restrict the following discussion to the most dominant contribution, and consider only the first term with a step function for (5.9).

The regimes where g_τ dominates are determined by the following conditions: For spins on opposite sides of the defects, it is necessary that $\tau > y/r$ and $y\tau + x\bar{\tau} < r$; for spins on the same side, the conditions are $\tau > \bar{y}/\bar{r}$ and $\bar{y}\tau + x\bar{\tau} < r$. In the former case, the first condition implies the second. So, if we fix $\sigma_{y_1,0}$, then the domain in which $\sigma_{y_2,x}$ can be located while satisfying the above condition is a wedge between the defect rows and the straight line

$$y_2 + \tau\bar{\tau}^{-1}x = y_1 \tag{5.11}$$

In the latter case, however, the condition $\tau > \bar{y}/\bar{r}$ does not imply $\bar{y}\tau + x\bar{\tau} < r$, and they have to be solved simultaneously. The solution gives a domain confined between the line of defects and the parabola

$$(\tau x - \bar{\tau} y_2 + \bar{\tau} y_1)^2 = 4\tau y_1(\bar{\tau} x + \tau y_2) \tag{5.12}$$

The shaded region in Fig. 3 shows half of these domains; the other half is its reflection with respect to the y axis. In the figure the angle $\phi = \arcsin \tau$.

The defect strength affects the correlations also in the form of prefactors that are orientation dependent as well. In Fig. 4, this is shown for the case where the spins are on opposite sides in the low-temperature regime. The abscissa is $\sin\theta = y/r$ and the ordinate is τ. The broken line separates the two regimes where g_τ dominates (above) and where the term with f_τ dominates (below). The values given for each curve is the dominant term in the net spin correlation normalized by the bulk correlation below the broken line, and by the factor $\exp(-r - y\tau - x\bar{\tau})/(2\pi r)^{1/2}$ above the broken line. The normalizer used in the latter case is an Ornstein–Zernike-type behavior modified by a special type of exponential decay peculiar to linear defect systems.

APPENDIX A. SOME DEFINITIONS

This Appendix gives the functions appearing in the text.

The integrand in the matrix A_0 in (2.2):

$$C(\phi) = C(-\phi)^{-1} = C^*(\phi)^{-1}$$
$$= [(1-\alpha_1 e^{i\phi})(1-\alpha_2 e^{-i\phi})]^{1/2}$$
$$\times [(1-\alpha_2 e^{i\phi})(1-\alpha_1 e^{-i\phi})]^{-1/2} \tag{A.1}$$

where

$$\begin{aligned} \alpha_1 &= z_1(1-z_2)/(1+z_2) < 1 \\ \alpha_2 &= z_1^{-1}(1-z_2)/(1+z_2) \gtrless 1 \qquad \text{for} \quad T \gtrless T_c \\ z_j &= \tanh(E_j/kT) \end{aligned} \tag{A.2}$$

The canonical factorization of $C(\phi)$:

$$C_+(\phi) = C_-(-\phi)^{-1} = (1-\alpha_1 e^{i\phi})^{1/2}(1-\alpha_2 e^{i\phi})^{-1/2} \tag{A.3}$$

The functions X^s_{pq} in the definition of A_{pq} in (2.6):

1. For spins on different sides of the defect, $l < 0 < m$,

$$X^s_{pq} = \begin{cases} U\exp[-(2m_p-1)\Gamma], & s=1,\ p=q \\ W\exp[-(m-l-1)\Gamma + i(p-q)\,n\phi], & s=1,\ p\neq q \\ 0 & s=2 \end{cases} \tag{A.4a}$$

2. For spins on the same side of the defect, $0 < l < m$,

$$X^s_{pq} = \begin{cases} U\exp[-(2m_p-1)\Gamma], & s=1,\ p=q \\ \exp[-(m-l)\Gamma + i(p-q)\,n\phi], & s=1,\ p\neq q \\ 0 & s=2,\ p=q \\ U\exp[-(m+l-1)\Gamma + i(p-q)\,n\phi], & s=2,\ p\neq q \end{cases} \tag{A.4b}$$

where

$$W(\phi) = W(-\phi) = (1-|\kappa|^2)(\kappa+C)^{-1}(\kappa^*+C^*)^{-1} \tag{A.5}$$

$$U(\phi) = -U(-\phi) = (\kappa C^* - \kappa^* C)(\kappa+C)^{-1}(\kappa^*+C^*)^{-1} \tag{A.6}$$

$$\Gamma(\phi) = \Gamma(-\phi) = \operatorname{arsinh}\{(1-z_2^2)[2z_2(1-z_1^2)]^{-1}$$
$$\times [(1-\alpha_1 e^{i\phi})(1-\alpha_2^{-1}e^{-i\phi})$$
$$\times (1-\alpha_2^{-1}e^{i\phi})(1-\alpha_1 e^{-i\phi})]^{1/2}\} \tag{A.7}$$

The defect strength κ:

$$\begin{aligned}
\kappa &= \tanh(\rho)e^{i\phi} \\
\cosh(2\rho) &= c_1 c_2 - s_1 s_2 \cos(\phi) \\
\sinh(2\rho)e^{i\phi} &= s_1 c_2 - c_1 s_2 \cos(\phi) + is_2 \sin(\phi) \\
c_1 &= \cosh[2(2E_1' - E_1)/kT], \quad c_2 = \cosh(2E_2'^*/kT)
\end{aligned} \tag{A.8}$$

and s_1 and s_2 are the corresponding hyperbolic sine functions.

In the scaling limit,

$$\begin{aligned}
\alpha_2 &\to 1 + \mathrm{sgn}(T - T_c)\xi_h^{-1} \\
\kappa &\to \kappa_c = \tanh[(2E_1' - E_1 - E_2'^*)/kT_c]
\end{aligned} \tag{A.9}$$

Define

$$\tau = \mathrm{sgn}(T - T_c)\tanh[2(2E_1' - E_1 - E_2'^*)/kT_c] \tag{A.10}$$

and scale the variable ϕ by ξ_h. Then, as $T \to T_c$,

$$\begin{aligned}
W(\phi) &\to W_c(w) = (1-\tau^2)^{1/2}(1+w^2)^{1/2}[(1+w^2)^{1/2} - \tau]^{-1} \\
U(\phi) &\to U_c(w) = -i\tau\,\mathrm{sgn}(T - T_c)\, w[(1+w^2)^{1/2} - \tau]^{-1} \\
\Gamma(\phi) &\to \Gamma_c(w)/\xi_v = (1+w^2)^{1/2}/\xi_v
\end{aligned} \tag{A.11}$$

The function Y^s_{pq} in (4.8) is the scaling limit of X^s_{pq}, $X^s_{pq}(\phi) \to Y^s_{pq}(w)$:

1. For spins on different sides of the defect, $l < 0 < m$,

$$Y^s_{pq} = \begin{cases} U_c \exp(-2y_p\Gamma_c), & s=1,\ p=q \\ W_c \exp[-y\Gamma_c + i(p-q)xw], & s=1,\ p\neq q \\ 0 & s=2 \end{cases} \tag{A.12a}$$

2. For spins on the same side of the defect, $0 < l < m$,

$$Y^s_{pq} = \begin{cases} U_c \exp(-2y_p\Gamma_c), & s=1,\ p=q \\ \exp[-y\Gamma_c + i(p-q)xw], & s=1,\ p\neq q \\ 0 & s=2,\ p=q \\ U_c \exp[-\bar{y}\Gamma_c + i(p-q)xw], & s=2,\ p\neq q \end{cases} \tag{A.12b}$$

APPENDIX B. ASYMPTOTIC ANALYSIS

In this Appendix, we calculate the leading orders of the spin–spin correlation function.

For large separations, Eq. (4.8) is approximated by the first few terms of the series,

$$\langle \delta\sigma_{l,0}\,\delta\sigma_{m,n}\rangle \approx \sum_{\{p_j,s_j\}}' \left[\frac{1}{4}\left(\frac{-i}{2\pi}\right)^2 \iint_{-\infty}^{\infty} dw_1\,dw_2 \right.$$
$$\times L_{p_1p_2}^{s_1s_2}(w_1,w_2)\,L_{p_2p_1}^{s_2s_1}(w_2,w_1)$$
$$+\frac{1}{6}\left(\frac{-i}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} dw_1\,dw_2\,dw_3\,L_{p_1p_2}^{s_1s_2}(w_1,w_2)$$
$$\left.\times L_{p_2p_3}^{s_2s_3}(w_2,w_3)\,L_{p_3p_1}^{s_3s_1}(w_3,w_1)\right] \tag{B.1a}$$

$$L_{pq}^{st}(w,\rho)=\frac{Y_{pq}^{t}(\rho)}{(1+w^2)^{1/2}}\,\frac{(1+w^2)^{1/2}-(-1)^{p(s+t)}\,(1+\rho^2)^{1/2}}{\omega-\rho+i\varepsilon} \tag{B.1b}$$

For $l<0<m$, the expression is simpler:

$$\langle \delta\sigma_{l,0}\,\delta\sigma_{m,n}\rangle \approx I \equiv \frac{1}{2}\,\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2$$
$$\times L_{12}^{11}(w_1,w_2)\,L_{21}^{11}(w_2,w_1) \tag{B.2}$$

Substitute for the integrand from (B.1b), (A.12a), and (A.11), and let $w_1 \to -w_1$; we have

$$I=\frac{1-\tau^2}{8\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2 \prod_{k=1}^{2}\left(\frac{\exp(-yw_k'-ixw_k)}{w_k'-\tau}\right)\frac{(w_1-w_2)^2}{(w_1'+w_2')^2} \tag{B.3}$$

where $w'=(1+w^2)^{1/2}$. (This notation is used throughout Appendix B.)

As $r=(x^2+y^2)^{1/2}\to\infty$, the integral is dominated by the contribution near the saddle point,

$$w_s=-ix/r\equiv -i\cos\theta \tag{B.4}$$

We deform the contours of integration form the real axis so that they pass through the saddle point on the negative imaginary axis. Because the denominator in the integrand can vanish, there is a contribution from the pole at

$$w_p=-i(1-\tau^2)^{1/2} \tag{B.5}$$

if w_s lies below w_p on the imaginary axis, i.e., if $\tau>y/r$.

Naively, the integral will consist of three terms: one from $w_1\sim w_s$ and $w_2\sim w_s$ (denote this term as I_s), one from $w_1(w_2)\sim w_s$ and $w_2(w_1)\sim w_p$

(denote as I_p), and one from $w_1 \sim w_p$ and $w_2 \sim w_p$. However, the last term vanishes because the residues is zero. So we write

$$I = I_s + \Theta(\tau - y/r)\, I_p \tag{B.6}$$

The pure saddle-point contribution can be expressed as

$$I_s = \frac{1-\tau^2}{8\pi^2} \iint_{-\infty}^{\infty} dw_1\, dw_2 \times \prod_{k=1}^{2} \left(\frac{\exp[-r - r\Delta_k^2 (2\sin^2\theta)^{-1}]}{\sin\theta - \tau} \right) \frac{2\Delta_1^2}{(2\sin\theta)^2} \tag{B.7}$$

where $\Delta_k = w_k - w_s$.

Equation (B.7) is obtained by expanding the integrand of (B.3) in powers of Δ_1 and Δ_2, retaining the lowest order term. It is easy to evaluate the Gaussian integrals above, and we find

$$I_s = (1-\tau^2) \left(\frac{y}{r\tau - y} \right)^2 \frac{e^{-2r}}{8\pi r^2} \tag{B.8}$$

The mixed saddle-point-pole contribution is

$$\begin{aligned} I_p &= 2(-2\pi i) \frac{1-\tau^2}{8\pi^2} \frac{\exp(-y\tau - x\bar{\tau})}{w_p (1+w_p^2)^{-1/2}} \\ &\quad \times \int_{-\infty}^{\infty} dw_2 \left(\frac{\exp[-r - r\Delta_2^2 (2\sin^2\theta)^{-1}]}{\sin\theta - \tau} \right) \frac{(w_p - w_s)^2}{(\tau + \sin\theta)^2} \\ &= \tau\bar{\tau} \frac{y}{r\tau - y} \left(\frac{r\bar{\tau} - x}{r\tau + y} \right)^2 \frac{\exp(-r - y\tau - x\bar{\tau})}{(2\pi r)^{1/2}} \end{aligned} \tag{B.9}$$

Substituting the above in (B.6) and (B.2) yields the formulas (5.6) and (5.7).

The situation for the case where the two spins are on the same side $(0 < l < m)$ is less clear. Which term in (B.1) dominates asymptotically depends very much on the relative magnitudes of x, y_1, and y_2. Therefore we will treat the double integrals and one term from the triple integrals as equally important in the following analysis.

There are four integrals to be evaluated asymptotically for the case where the two spins are on the same side:

$$
\begin{aligned}
\langle \delta\sigma_{l,0}\,\delta\sigma_{m,n}\rangle &\approx J_0+J_1+J_2+J_3\\
J_0 &\equiv \frac{1}{8\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2\,L_{12}^{11}(w_1,w_2)\,L_{21}^{11}(w_2,w_1)\\
J_1 &\equiv \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2\,L_{22}^{12}(w_1,w_2)\,L_{21}^{21}(w_2,w_1)\\
J_2 &\equiv \frac{1}{8\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2\,L_{12}^{22}(w_1,w_2)\,L_{21}^{22}(w_2,w_1)\\
J_3 &\equiv \frac{i}{8\pi^3}\iiint_{-\infty}^{\infty} dw_1\,dw_2\,dw_3\,L_{12}^{11}(w_3,w_2)\\
&\quad\times L_{21}^{11}(w_2,w_1)\,L_{11}^{11}(w_1,w_3)
\end{aligned}
\tag{B.10}
$$

Substituting the L_{pq}^{st} from (B.1), (A.12b), and (A.11) and changing w_1 to $-w_1$, we find for the above integrals

$$
\begin{aligned}
J_0 &= \frac{1}{8\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2 \prod_{k=1}^{2}\frac{\exp(-yw_k'-ixw_k)}{w_k'}\frac{(w_1-w_2)^2}{(w_1'+w_2')^2}\\
J_1 &= \frac{i\tau}{4\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2\,\frac{\exp(-yw_1'-ixw_1)\exp(-\bar{y}w_2'-ixw_2)}{w_1'w_2'(w_2'-\tau)}\,w_2\,\frac{w_1-w_2}{w_1+w_2-i\varepsilon}\\
J_2 &= \frac{-\tau^2}{8\pi^2}\iint_{-\infty}^{\infty} dw_1\,dw_2 \prod_{k=1}^{2}\frac{w_k\exp(-\bar{y}w_k'-ixw_k)}{w_k'(w_k'-\tau)}\frac{(w_1-w_2)^2}{(w_1'+w_2')^2}\\
J_3 &= \frac{-\tau}{8\pi^3}\iiint_{-\infty}^{\infty} dw_1\,dw_2\,dw_3\,\frac{w_3\exp(-2y_1w_3')}{w_3'(w_3'-\tau)}\\
&\quad\times\prod_{k=1}^{2}\left(\frac{\exp(-yw_k'-ixw_k)}{w_k'}\right)\frac{(w_3+w_2)(w_2-w_1)(w_3-w_1)}{(w_1'+w_2')(w_2'+w_3')(w_3'+w_1')}
\end{aligned}
\tag{B.11}
$$

The dominant contributions for the integrals (B.11) are computed in the same way as in (B.3): J_0 is dominated by the contribution from the saddle point w_s of (B.4); J_1 has two terms, one from $w_1 \sim w_s$ and $w_2 \sim \bar{w}_s$, and one from $w_1 \sim w_s$ and $w_2 \sim w_p$; J_2 also has two terms coming from $w_1 \sim \bar{w}_s$ and $w_2 \sim \bar{w}_s$, and from $w_1(w_2) \sim \bar{w}_s$ and $w_2(w_1) \sim w_p$; and J_3 comes mainly from $w_1 \sim w_s$, $w_2 \sim w_s$, and $w_3 \sim 0$. The image saddle point $\bar{w}_s$ is similar to w_s,

$$
\bar{w}_s = -ix/\bar{r} \tag{B.12}
$$

Proceeding as with I_s and I_p before, we find

$$J_0 \approx \frac{\exp(-2r)}{8\pi r^2}$$

$$J_1 \approx -\frac{\tau x}{\bar{r}\tau - \bar{y}} \frac{\bar{r} - r}{\bar{r} + r} \frac{\exp(-r - \bar{r})}{2\pi(r\bar{r})^{1/2}} - \Theta\left(\tau - \frac{\bar{y}}{\bar{r}}\right) \tau \frac{r\bar{\tau} - x}{r\bar{\tau} + x} \frac{\exp(-r - \bar{y}\tau - x\bar{\tau})}{(2\pi r)^{1/2}}$$

$$J_2 \approx \tau^2 \left(\frac{x}{\bar{r}\tau - \bar{y}}\right)^2 \frac{\exp(-2\bar{r})}{8\pi\bar{r}^2} + \Theta\left(\tau - \frac{\bar{y}}{\bar{r}}\right) \frac{\tau^2 x}{\bar{r}\tau - \bar{y}} \left(\frac{\bar{r}\bar{\tau} - x}{\bar{r}\tau + y}\right)^2 \frac{\exp(-\bar{r} - \bar{y}\tau - x\bar{\tau})}{(2\pi\bar{r})^{1/2}}$$

$$J_3 \approx -\frac{\tau}{1 - \tau} \frac{ry}{(r + y)^2} \frac{\exp(-2r - 2y_1)}{(4\pi y_1)^{3/2} r^2} \tag{B.13}$$

The above give (5.8) and (5.9).

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